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A survey is presented of the author's theory of 3-part splittings and to its advantages over the "classical" 2-part splittings (eg. Gauss-Seidel, SOR). In particular analysis is made for the case when the 2-part iteration matrix B has (a) real spectrum, and (b) pure complex spectrum. M. Neumann's work on 3-part splittings for rectangular matrices is part of the section of open problems.

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Improved convergence for linear systems using three-part splittings*

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Introduction

We seek the solution vector x in Hilbert space H for the linear system $Ax = y_o$, where $y_o \in H$ is given in advance and A is a bounded linear invertible operator on H. We shall compare convergence of two-part splittings (i.e., linear stationary methods of first degree) with convergence of three-part splittings (i.e., linear stationary methods of second degree). In fact, we write the two-part and three-part splittings of A as follows:

$$A = A_1 + A_2' \tag{1.1a}$$

and $$A = A_1 + A_2 + A_3, \tag{1.1b}$$

where A_1, A_2', A_2 and A_3 are all bounded linear operators on H and A_1^{-1} is presumed easy to compute (for example, if H is finite-dimensional, then A_1 might be represented by a diagonal matrix). We are now ready to define the respectively induced two-part and three-part sequences, $\{x_k'\}$ and $\{x_k\}$.

Definition 1.1. Given $A = A_1 + A_2'$ where $A_2' = A_2 + A_3$, with fixed $y_o \in H$. Then for any $x_o = x_o' \in H$ the two-part sequence

$$\{x_o, x_1', x_2', \ldots, x_k', \ldots\}$$

is defined iteratively by

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$$A_1x'_{k+1} + A_2'x'_k = y_o \; , \; k = 0,1,2,\ldots \tag{1.2a}$$

The three-part sequence $\{x_k\}$ is defined iteratively by

$$A_1x_{k+2} + A_2x_{k+1} + A_3x_k = y_o \; , \; k = 0,1,2,\ldots \tag{1.2b}$$

Note that if either the two-part sequence $\{x_k'\}$ or the three-part sequence $\{x_k\}$ converges, then since A is invertible, convergence is necessarily to the solution vector x where $Ax = y_o$. We shall comment on the case where A is singular, at the end of this paper. Now if A is a non-singular matrix on finite-dimensional H, then consider A may be decomposed as

$$A = D + E + F,$$

where D is the diagonal part of A, E is the (lower triangular) matrix whose entries match those of A below the main diagonal and F is the corresponding upper triangular matrix taken from the entries of A above the main diagonal. The two-part splitting (1.1a) then yields the well-known schemes

(a) Choosing $A_1 = D$, we have the Jacobi iteration scheme,

(b) Choosing $A_1 = D + E$, we have the Gauss-Seidel iteration scheme, (1.2c)

(c) Choosing $A_1 = \frac{1}{\omega}D + E$, with $0 < \omega < 2$, we have the successive overrelaxation (SOR) method.

Three-part splittings are given mention in [7] and [9]. Also, in a paper of J. D. P. Donnelly, a theorem on chaotic relaxation is proven [5, Theorem 2.1]. This result is recaptured (and generalized) in the setting of three-part splittings, cf. [2, Theorem 5.4].

How are the two-part and three-part splittings related? Observe that (1.2b) is equivalent to

$$\underset{\substack{\uparrow \\ Z_{k+1}}}{\begin{bmatrix} x_{k+2} \\ x_{k+1} \end{bmatrix}} = \underset{\substack{\uparrow \\ B_2}}{\begin{bmatrix} -A_1^{-1}A_2 & -A_1^{-1}A_3 \\ I & 0 \end{bmatrix}} \underset{\substack{\uparrow \\ Z_k}}{\begin{bmatrix} x_{k+1} \\ x_k \end{bmatrix}} + \underset{\substack{\uparrow \\ b}}{\begin{bmatrix} A_1^{-1}y_o \\ 0 \end{bmatrix}}, \tag{1.3}$$

where $x_k \to x$ in H if and only if $Z_k \to Z = \begin{bmatrix} x \\ x \end{bmatrix}$ in $H \oplus H$. Moreover, since (1.3) is the result of a two-part splitting on $H \oplus H$,

$$(Z_k - Z) = B_2^{\,k}(Z_o - Z). \tag{1.4}$$

Compare (1.4) to the condition resulting from the two-part splitting (1.1a), viz.,

$$(x_k{}' - x) = B^k(x_o - x),\ B = -A^{-1}A_2{}'. \tag{1.5}$$

In other words, as $B = -A_1{}'A_2$ is the transition (or iteration) operator for the two-part splitting (1.1a), so is B_2 (given in (1.3)) the transition matrix operator induced by the three-part splitting (1.1b). The importance of these remarks is that a sufficient condition for the convergence of sequences $\{x_k{}'\}$ of (1.2a) or $\{x_k\}$ of (1.2b) is that the spectral radius of the respective transition operators $\rho(B)$, and $\rho(B_2)$, be less one (this condition is necessary for finite-dimensional H). We are ready for the following definition:

Definition 1.2. Given the sequences $\{x_k{}'\}$ of (1.2a) and $\{x_k\}$ of (1.2b), then their respective asymptotic convergence rates, R' and R, are given by

$$R' = -\log_{10}\rho(B),$$
$$R = -\log_{10}\rho(B_2),$$

where $B = -A_1^{-1}A_2'$ from (1.1a) and B_2 is given in (1.3).

Remark. We indicate base ten in the definition above only as a convenience since $1/R$ will then indicate, roughly, the number of iterations or "hits" which will produce one more decimal place of accuracy (cf. [8, page 63]).

We conclude this section with an explicit statement of the goals of this paper. Given $A_1x_{k+2} + A_2x_{k+1} + A_3x_k = y_o$ with A_1 fixed (and presumed easy to invert). Then we seek to

1) Explicitly construct A_3 (and hence, A_2). (Theorem 2.1, display (2.2), Hypothesis 3).
2) Explicitly compute $\rho(B_2)$, and to
3) Compare with $\rho(B)$ (noting that $A = A_1 + A_2' = A_1(I-B)$). (Theorem 3.1, display (3.4) if $\rho(B)$ is real; Theorem 3.3 if $\rho(B)$ is imaginary.)

Thus, if for a fixed two-part splitting $A = A_1 + A_2'$, we can define that A_3 in the three-part splitting $A = A_1 + A_2 + A_3$ where $\rho(B_2) < 1$ and $\rho(B_2) < \rho(B)$, then we will have established the greater convergence rate of $\{x_k\}$ in (1.2b) over that of $\{x_k'\}$ in (1.2a).

2. Relating $\sigma(B)$ and $\sigma(B_2)$ via analytic ϕ.

Recall that we are attempting to compare the spectral radius $\rho(B)$ of iteration operator B (see 1.5)) with the spectral radius $\rho(B_2)$ of the iteration matrix operator B_2 (see (1.3), (1.4)). In the following theorem, we assume that some bounds on $\sigma(B)$, the spectrum of B, are known (Theorem 2.1, Hypothesis 1). Then, A_3 is explicitly constructed as a certain analytic operator function of B (Hypothesis 3). This automatically fixes A_2 and hence, defines B_2 in (1.3). The theorem characterizes $\sigma(B_2)$ in terms of $\sigma(B)$, depending on the analytic $\phi(\cdot)$ chosen (cf. (2.3)).

<u>Theorem 2.1</u> ([4, Th. 3.1]). Given invertible $A = A_1 + (A_2 + A_3)$ where $-B = A_1^{-1}(A_2 + A_3)$. Suppose

1. $\sigma(-B)$ lies inside the cardioid (cf. Figure 1)

$$\mathcal{C} = \{2Z(1 + \mathrm{Re}(Z)) - 1 : Z = e^{i\theta}\} . \tag{2.1}$$

2. $\phi(\cdot)$ is a complex analtyic function whose domain of definition contains $\sigma(-B)$, and does not assume the value -1 on $\sigma(-B)$.

3.
$$A_3 = -A_1\phi(-B)\frac{(\phi(-B) + B)}{(\phi(-B) + I)}, \tag{2.2}$$

where $\phi(-B)$ is the corresponding analytic operator-valued function of -B, and I is the identity operator. Then with B_2 given in (1.3),

$$-\sigma(B_2) = \sigma(\phi(B)) \cup \sigma\left(\frac{\phi(-B) + B}{\phi(-B) + I}\right) . \tag{2.3}$$

<u>Proof</u>. Use $\phi(-B)$ to define U,V and A_3 as follows:

$$U = \phi(-B)$$

$$V = -\left(\frac{\phi(-B) + B}{\phi(-B) + I}\right)$$

$$A_3 = A_1 UV .$$

Then verify the identity

$$-B = U + V + UV .$$

Note that from (1.3),

$$B_2 = \begin{bmatrix} U+V & UV \\ -I & 0 \end{bmatrix}.$$

Let $W = \begin{bmatrix} I & -V \\ 0 & I \end{bmatrix}$ so that $W^{-1}B_2W$ (which has the same spectrum as B_2) has form

$$W^{-1}B_2W = \begin{bmatrix} U & 0 \\ -I & V \end{bmatrix}$$

$$\Rightarrow \sigma(B_2) = \sigma(W^{-1}B_2W) = \sigma(U) \cup \sigma(V) .$$

The conclusion now follows, and the proof is done. ■

Remark. Why does the cardiod $\mathcal{C}$ (Figure 1) enter into the theorem in (2.1)? The reason is that we require that $\rho(B_2) < 1$ so as to guarantee convergence. It is shown in [2, pg. 335] that it is just this condition, $\rho(B_2) < 1$, that implies $\sigma(-B)$ is a subset of the interior of cardioid $\mathcal{C}$. In the special case where $\phi(\cdot)$ is a constant, however, bounding $\sigma(-B)$ by $\mathcal{C}$ is too crude. In fact, we note that for constant $\phi(Z) = s \neq -1$, we have

$$\rho(B_2) < 1 \Rightarrow \sigma(-B) \in \{Z:|Z-1| < 2\} , \qquad (2.4)$$

the proof of which is in [4, Prop. 3.3].

3. The case $\phi(Z)$ is constant.

Henceforth, we assume that analytic $\phi(Z) = s \neq -1$ for fixed complex s and all complex Z. In operator terms, $\phi(B) = sI$, for all operators B on H. One immediate simplication is that A_3 is easy to compute, given the representation $A = A_1 + A_2 + A_3$ with A_3 defined in (2.2) as Hypothesis 3 of Theorem 2.1. In fact, equation (1.2b) for the three-part splitting (using (2.2) with $\phi(Z) = s$) reduces to

$$
\begin{aligned}
x_{k+2} = -A_1^{-1}A\left(\frac{1}{s+1}\,x_{k+1} + \frac{s}{s+1}\,x_k\right) & \\
+ ((1-s)x_{k+1} + sx_k) + A_1^{-1}y_o\,, & \qquad (3.1)
\end{aligned}
$$

where complex $s \neq -1$, $A = A_1 + A_2 + A_3 = A_1(I-B) = A_1 + A_2'$.

<u>Remark</u>. If complex s is chosen to be zero, then (3.1), the equivalent to the three-part expression (1.2b), reduces to the two-part splitting (1.2a).

We give an analysis of three-part convergence versus two-part convergence for $A = A_1(I-B)$ in the following two case of the iteration operator B, viz., when the spectrum of B, $\sigma(B)$, is real, and when it is pure imaginary.

<u>3.1 Case (A): $A = A_1(I-B)$, $\sigma(B)$ is real ($\phi(Z) = s$ = constant)</u>.

We have already noted that if $\rho(B_2) < 1$, for B_2 given in (1.3), then, necessarily, $\sigma(-B)$ is a subset of the interior of cardioid $\mathcal{C}$ of (2.1), and if analytic ϕ is constant, then $\sigma(-B)$ is a subset of the open disc in the complex plane centered at real 1, having radius 2 (cf. (2.4)). If, moreover, $\sigma(B)$ is real, then this reduces to the condition that there exists real $\alpha, \beta \in \sigma(-B)$ such that for all $\lambda \in \sigma(-B)$, $-1 < \alpha \le \lambda \le \beta < 3$. The next theorem responds to the following:

1) What is the optimal real s, call it s_o, we may assign to $\phi(-B) = sI$ so as to give optimal $\rho(B_2)$, the minimal spectral radius of B_2 of (1.3)? Note that B_2 depends on the choice of A_3 from (1.3), and A_3, in turn, depends on $\phi(-B) = sI$ from (2.2).

2) Given optimal s_o, what is the explicit value of the optimal, or minimal $\rho(B_2)$ as a function of the spectral radius $\rho(B)$?

Here is that theorem now.

<u>Theorem 3.1</u>. Given $A = A_1(I-B)$, $\sigma(B)$ real (where A_1^{-1} is easy to compute). Suppose $\alpha, \beta \in \sigma(-B)$ where fore all $\lambda \in \sigma(-B)$,

$$-1 < \alpha \le \lambda \le \beta < 3. \tag{3.2}$$

Let m denote the midpoint of $\sigma(-B)$, i.e.,

$$m = \frac{\alpha + \beta}{2} . \tag{3.3}$$

Then, according to whether $m > 0$ or $m < 0$, we have

If $m \le -1 + \sqrt{1 + \rho(B)}$	If $m \ge -1 + \sqrt{1 + \rho(B)}$	
Then $s_o = m$ $\rho_o(B_2) = \frac{\rho(B) - m}{1 + m}$	Then $s_o = -1 + \sqrt{1 + \rho(B)}$ $\rho_o(B_2) = -1 + \sqrt{1 + \rho(B)}$	$m > 0$ ($\rho(B) = \beta$)

(3.4)

If $m \ge -1 + \sqrt{1 - \rho(B)}$	If $m \le -1 + \sqrt{1 - \rho(B)}$	
Then $s_o = m$ $\rho_o(B_2) = \frac{\rho(B) + m}{1 + m}$	Then $s_o = -1 + \sqrt{1 - \rho(B)}$ $\rho_o(B_2) = 1 - \sqrt{1 - \rho(B)}$	$m < 0$ $\rho(B) = -\alpha$

By setting $\phi(-B) = s_o I$ in (2.2), we construct that B_2 in (1.3) whose spectral radius is minimal relative to all $\phi(-B) = sI$, as s varies over all real s.

Moreover, in all four cases in the table, above, three-part convergence is better than two-part convergence in the sense that

$$\begin{aligned} &\rho(B_2) < \rho(B), \text{ or} \\ &R > R' \text{ (cf. Def. 1.2).} \end{aligned} \tag{3.5}$$

On the proof of Theorem 3.1. Complete details appear in [4, section 5]. The proof depends on several technical lemmas whose end objective is to establish that $\rho(B_2) = f \vee g$, the maximum of a pair of functions f and g, each defined for real s (recall that $\phi(-B) = sI$). The optimal (smallest) value of $\rho(B_2)$, which we denote $\rho_o(B_2)$, is computed as the minimum of $f \vee g$. These are the values given for $\rho_o(B_2)$ in the table above. ■

Remark. Observe that in case the midpoint $m < 0$, then it is automatic that $\rho(B) < 1$, so that in the table (3.4), the expression $\sqrt{1 - \rho(B)}$ is always well defined. Also, when $m \neq 0$, we produce an s_o where $\rho_o(B_2)$ in all four categories is always less than $\rho(B)$. This is easy to confirm directly from the table (3.4).

Example 3.2. Consider the 3x3 matrix A and its decompositions

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= A_1(I - \underbrace{\mathcal{L}_\omega}_{B}) \qquad \text{(SOR decomposition)} \tag{3.6}$$

$$= I - \underbrace{\begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}} \qquad \text{(Jacobi decomposition)}$$

where for all ω, $0 < \omega < 2$, $\mathcal{L}_\omega$, the SOR iteration matrix (cf. (1.2c)) given by

$$B = -\mathcal{L}_\omega = \begin{bmatrix} \omega-1 & 0 & \omega \\ -\omega(\omega-1) & \omega-1 & -\omega^2 \\ \omega^2(\omega-1) & -\omega(\omega-1) & \omega^3+\omega+1 \end{bmatrix} . \tag{3.7}$$

Note that (3.6) presents two two-part splittings for A, viz., the SOR decomposition with iteration matrix, $B = \mathcal{L}_\omega$, and the point-Jacobi decomposition with iteration matrix J. Note that J is consistently ordered, cyclic of index three with eigenvalues equal to the cube roots of -1. Under these conditions, we may use $\sigma(J)$ the spectrum of J, to find $\sigma(\mathcal{L}_\omega)$, the spectrum of $\mathcal{L}_\omega$ [8, Theorem 4.3]. In fact, let ω, $0 < \omega < 2$ define $\mathcal{L}_\omega$ in (3.7). Let J be given as in (3.6). Then

$$\text{for all } \lambda \in \sigma(J), \text{ we have } \mu \in \sigma(\mathcal{L}_\omega) \text{ if and only if}$$

$$(\lambda + \omega - 1)^3 + \lambda^2\omega^3 = 0 . \tag{3.8}$$

Let us select

$$\omega = \omega_o = 0.897107 \tag{3.9}$$

which, from (3.8) yields the spectrum of the SOR iteration matrix $\mathcal{L}_{\omega_o}$, i.e., for $\omega_o = 0.897107$,

$$\sigma(-\mathcal{L}_{\omega_o}) = \{-0.026473,\ 0.211778,\ 0.211791\},$$

so that the hypotheses (3.2) and (3.3) of Theorem 3.1 write themselves as follows:

$$\begin{aligned} \alpha &= -0.026473 \\ \beta &= \ \ 0.211791 = \rho(\mathcal{L}_{\omega_o} = B) \\ m &= \ \ 0.092659 . \end{aligned} \tag{3.10}$$

From this, the SOR convergence rate for the two-part splitting of (3.6) is

$$R' = -\log_{10}(\rho(\mathcal{L}_{\omega_o})) \approx 0.674. \tag{3.11}$$

Note. By trial and error, it seems the ω_o of (3.9) is optimal, i.e., produces the smallest $\rho(\mathcal{L}_\omega)$. In any case, the fact that the midpoint m of (3.10) is not zero, guarantees that improved convergence must result in passing to the three-part splitting. In particular, from (3.10) $m^2 < -\alpha$ and $m > 0$. From (3.4) in Theorem 3.1, we see that optimal s_o is

$$s_o = m \approx 0.092659,$$

while optimal (smallest) spectral radius $\rho(B_2)$ is

$$\rho(B_2) = \frac{\rho(B) - m}{1 + m} \approx 0.109,$$

which defines the three-part convergence rate

$$R = -\log_{10}(\rho(B_2)) \approx 0.962 \tag{3.12}$$

(compare with (3.11)).

Finally, note that the three-part convergence rate (3.12) predicts (asymptotically) $1/R \approx 1$ iteration for each decimal place of accuracy, while the two-part SOR rare R' of (3.11) predicts $1/R' \approx 1.5$ iterations per additional decimal place of accuracy. We found in actual numerical examples with various values of y_o for $Ax = y_o$, and for various initial vectors x_o, that these asymptotic convergence rates manifested themselves after only seven or eight iterations.

3.2 Case (B): $A = A_1(I-B)$, $\sigma(B)$ is imaginary ($\phi(z) = s$ = constant

In the previous section ($\sigma(B)$ real), any two part splitting $A = A_1 + A_2 = A(I-B)$, whose iteration operator B had an unbalanced spectrum about the origin ($m \neq 0$) would always yield to improvement convergence rate by passing to the three-part splitting (Theorem 3.1). This section considers the case when $\sigma(B)$ is on the imaginary axis which proves to be far simpler than the real case. First, observe that since analytic $\phi(Z) \equiv s$ is constant, $\sigma(B)$, the spectrum of B lies inside the circle of radius two, centered at real 1 (cf. (2.4)). This means that all λ in for $\sigma(B)$ imaginary, we have $|\lambda| < \sqrt{3}$. The imaginary $\sigma(B)$ proves simpler than the real $\sigma(B)$ because whenever $\rho(B) < \sqrt{3}$ (so that the two part sequence $\{x_k'\}$ may diverge), the three-part splitting will always produce a convergent sequence $\{x_k\}$. Moreover, no restrictions on the imaginary midpoint apply. The following theorem states the details for passing to the optimal three-part splitting (1.2b) via construction of A_3 as per (2.2) for specifically defined constant $\phi(-B) = sI$.

Theorem 3.3. Given $A = A_1(I-B)$, $\sigma(B)$ imaginary (where A_1^{-1} is easy to compute). Suppose $\rho(B) < \sqrt{3}$. Then

case (a): If $\frac{\sqrt{5}-1}{2} \leq \rho(B)^2 < 3$ ($0.786 \leq \rho(B) < 1.732$) then the optimal $s = s_o$ is the unique real solution in $(-\rho(B), \rho(B))$ to the polynomial

$$s_o^4 + 2s_o^3 - \rho(B)^2 = 0 . \quad (0 \leq |s_o| < \rho(B))$$

Moreover, the optimal (smallest) $\rho(B_2)$ is then given by

$$\rho(B_2)_{optimal} = |s_o| .$$

<u>case (b)</u>: If $0 < \rho(B)^2 \le \frac{\sqrt{5}-1}{2}$ $(0 < \rho(B) \le 0.786)$, then the optimal $s = s_o$ is given by

$$s_o = \rho(B)^2,$$

in which case the optimal $\rho(B_2)$ is given by

$$\rho(B_2)_{optimal} = \frac{\rho(B)}{\sqrt{1 + \rho(B)^2}} .$$

For case (a) and case (b) above,

$$\rho(B_2) < 1, \text{ and } \rho(B_2) < \rho\ (B).$$

<u>On the proof of Theorem 3.3</u>. The complete proof requires the structure theorem of three-part splittings for constant $\phi(Z) = s$ [4, Theorem 4.2], which we do not present here. ∎

<u>Remark</u>. Observe from the statement of Theorem 3.3, case (a), that even if $A = A_1(I - B)$, where $\rho(B)$ is near to unity (so that $\{x_k'\}$ converges slowly, if at all), then the three-part splitting guarantees a $\rho(B_2)$ very near to s_o where $s_o^4 + 2s_o^3 - 1 = 0$, i.e., $\rho(B_2) \approx 0.7167$ (so that $\{x_k\}$ converges). That is, although the two-part convergence rate $R' = \log_{10}(\rho(B)) \approx 0$, we have the three-part convergence rate $R = -\log_{10}(\rho(B_2)) \approx 0.14$. This fact holds regardless of the size of the system. To illustrate, consider the following example.

<u>Example 3.4</u>

The class $\mathcal{A}$ of n x n matrices, n varies, is defined by $\mathcal{A} = \{A(n): n = 1,2,3,\ldots\}$ where each A(n) is the tridiagonal n x n matrix

$$A(n) = \begin{bmatrix} 2 & 1 & & & 0 \\ -1 & 2 & 1 & & \\ & -1 & 2 & \ddots & \\ & & \ddots & \ddots & 1 \\ 0 & & & -1 & 2 \end{bmatrix} = 2I_n(I_n - B(n)) .$$

Accordingly, the n x n skew-symmetric iteration matrix B(n) for the two-part splitting $A(n) = 2I_n + A_2' = 2I_n(I_n - B(n))$ is

$$B(n) = \tfrac{1}{2} \begin{bmatrix} 0 & -1 & & & 0 \\ 1 & 0 & -1 & & \\ & 1 & 0 & \ddots & \\ & & \ddots & \ddots & -1 \\ 0 & & & 1 & 0 \end{bmatrix} .$$

Now the n-element, imaginary spectrum $\sigma(B(n))$, which depends on n, is

$$\sigma(B(n)) = \left\{ i \cos\left(\frac{k\pi}{n+1}\right) : k = 1,2,\ldots n \right\}$$

so that

$$\rho(B(n)) = \cos\left(\frac{\pi}{n+1}\right) \uparrow 1 \text{ as } n \uparrow \infty .$$

(See [1, page 380, Ex. 8, 9] for a related example.)

Now for $n \geq 4$, case (a) of Theorem (3.3) applies. That is, suppose $A(n)x = y_o$. Then as $n \to \infty$, the three-part splitting will produce a convergent sequence $\{x_k\}$, which eventually has every seventh term ($1/R \approx 7$) yielding one more decimal place of accuracy, while the two-part sequence $\{x_k'\}$ (given skew-symmetric iteration matrix B(n)) becomes slower and slower in its convergence.

Actual computations reveal that even when n is large, the asymptotic convergence rate prevails after only about a dozen iterations.

4. Some open questions

Recall that our investigation of three-part splittings (1.1b) for invertible operator A on H relied on the imbedding (1.3) as a two-part splitting on the larger space $H \oplus H$; for this reason, convergence rate analysis of $\{x_k\}$ versus that of $\{x_k'\}$ (cf. Def. 1.1) is effected by comparing the spectral radius of B_2 of (1.3) with that of B in (1.5).

Given fixed iteration operator B for $A = A_1(I-B)$ (equivalently, invertible A_1 is fixed) therefore, we constructed B_2 by first analytically defining $\phi(B)$ which, in turn, fixes A_3 (cf. (2.2) of Theorem 2.1), and hence, also fixes A_2 in the expression $A = A_1 + A_2 + A_3$.

This paper proceeds under two restrictions, namely that the analytic function ϕ is a constant s (in operator terms, $\phi B \equiv sI$), and that the operators $A = A_1 + A_2'$ and A_1 are both invertible; A_1^{-1} is supposedly easy to compute. Some work has been done in each of these areas where these two restrictions are relaxed. We briefly mention these results, while indicating paths of further exploration.

When analytic $\phi(\cdot)$ is not constant. In [2, p. 336], linear analytic $\phi(Z) = p_1z + p_2$ is explored under the assumption that $\sigma(-B)$ lies in the circle centered at 1, having radius two. This constraint on $\sigma(-B)$ is necessary if ϕ is given in advance as a constant function (see Remarks at the end of section 2).

In [3], non-constant ϕ is also studied. The main difficulty is in the computation of A_3 in (2.2) which involves the inverse of $(\phi(-B) + I)$; in [2], this entails the computation of $(-p_1B + (p_2 + 1)I)^{-1}$ which may be as difficult as the computation for A^{-1}, the inverse of the original operator A. (Of course, in the present paper, where $\phi(B) \equiv sI$, then $(\phi(-B) + I)^{-1}$ is just the scaler operator $(s + 1)^{-1}I$. Hence the questions:

4.1 <u>Given the operator A, and the invertible component A_1 (so that $A = A_1 + A_2'$), and given the explicit construction of A_3 for analytic ϕ in (2.2), what is the convergence rate of the three-part sequence $\{x_k\}$ defined by $A = A_1 + A_2 + A_3$, compared to that of the two-part sequence $\{x_k'\}$ defined by $A = A_1 + A_2'$ (necessarily, $A_2' = A_2 + A_3$), when analytic ϕ is not constant?</u>

4.2 <u>Moreover, under what circumstances, or for which ϕ, are the operator terms A_2 and A_3 easy to compute?</u> (More exactly, since from (1.2b), $x_{k+2} = -A_1^{-1}A_2x_{k+1} - A_1^{-1}A_3x_k + A_1^{-1}y_o$, we require easy computation only of the operator products $A_1^{-1}A_2$, $A_1^{-1}A_3$ and, of course, A_1^{-1}.

<u>When A is not invertible</u>. In a paper of Michael Neumann [6], three-part splittings are studied for non-invertible operators on finite-dimensional Hilbert space H, i.e., for rectangular matrices A. Given that A is m x n matrix (an operator sending n-space to m-space by left-hand multiplication on the columns), and given y_o in m-space, we are to find x in n-space such that $Ax = y_o$. Either a solution vector x exists, or else the system is not solvable, i.e., an approximate or best solution may be obtained. In the second instance, Neumann writes

$$A = A_1 + A_2'$$

where the splitting is <u>subproper</u>, i.e.,

$$rg(A) \subseteq rg(A_1) \text{ and } n(A) \supseteq n(A_1),$$

where rg(·) and n(·) indicate range and nullspace, respectively. If equality obtains, the splitting is <u>proper</u>. The idea then is to compose the iteration matrix

$$B = -A_1^+ A_2'$$

where A_1^+ is the Moore-Penrose generalized inverse of A_1. Thus, a counterpart to matrix B_2 of (1.3) is possible. Under these more general conditions, Neumann considers two cases, viz., when $A_1^+A_2'$ is or is not weakly convergent ($A_1^+A_2'$ is <u>weakly convergent</u> means the splitting $A = A_1 + A_2'$ is subproper, $\lambda \in (A_1^+A_2') \Rightarrow |\lambda| \leq 1$, and the Jordan blocks corresponding to $\lambda = 1$ are of order one).

<u>Case (a)</u> [6]. $-B = A_1^+A_2'$ is not weakly convergent: the constraint here is that $\sigma(-B) \subset \mathcal{C}^o \cup \{-1\}$, where $\mathcal{C}^o$ is the interior of the cardioid $\mathcal{C}$ of (2.1).

<u>Case (b)</u> $-B = A_1^+A_2'$ is weakly convergent. Then it is further assumed that $\sigma(-B) \subset \{Z: Re(Z) \geq 0\} \cup \{-1\}$.

Among many other things, Newmann exhibits certain analytic ϕ which effect A_3 in the construction of the generalized three-part splitting

$$A = A_1 + A_2 + A_3.$$

The question is raised, therefore

4.3 <u>Can specificially computed convergence rates be given for non-invertible matrices A under a three-part splitting, relative to rates under a two-part splitting?</u>

4.4 Can the results of Neumann be extended to the operator case for infinite-dimensional Hilbert space H?

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